

FROM H_v -RINGS TO H_v -FIELDS

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ABSTRACT. The general definition of a hyperfield is based on the fundamental relation defined on a general hyperring. The largest class of hyperstructures is the one introduced in 1990, called H_v -structures which satisfy the weak properties. We present a general way to construct an H_v -field from a given H_v -ring and we study some properties of them. Moreover we present some recent applications of the above hyperstructures.

1. Introduction

The main object of this paper is the class of hyperstructures called H_v -structures introduced in 1990 [13], which satisfy the weak axioms where the non-empty intersection replaces the equality. Some basic definitions are the following:

Algebraic hyperstructure is called any set H equipped with at least one hyperoperation (abbreviation: *hyperoperation=hope*) $\cdot : H \times H \rightarrow P(H) - \{\emptyset\}$. We abbreviate by *WASS* the *weak associativity*: $(xy)z \cap x(yz) \neq \emptyset, \forall x, y, z \in H$ and by *COW* the *weak commutativity*: $xy \cap yx \neq \emptyset, \forall x, y \in H$. The hyperstructure $(H, \cdot)$ is called H_v -semigroup if it is *WASS*, it is called H_v -group if it is reproductive H_v -semigroup, i.e., $xH = Hx = H, \forall x \in H$.

Motivation. In the classical theory the quotient of a group with respect to an invariant subgroup is a group. F. Marty from 1934, states that, the quotient of a group with respect to any subgroup is a hypergroup. Finally, the quotient of a group with respect to any partition (or equivalently to any equivalence relation) is an H_v -group. This is the motivation to introduce the H_v -structures [13][16].

In an H_v -semigroup the powers of an element $h \in H$ are defined as follows: $h^1 = \{h\}, h^2 = h \cdot h, \dots, h^n = h \circ h \circ \dots \circ h$, where $(\circ)$ denotes the *n-ary circle hope*, i.e. take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An H_v -semigroup $(H, \cdot)$ is called *cyclic of period s*, if there exists an element h , called *generator*, and a natural number s , the minimum one, such that $H = h^1 \cup h^2 \cup \dots \cup h^s$. Analogously the cyclicity for the infinite period

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is defined [13]. If there is an element h and a natural number s , the minimum one, such that $H = h^s$, then $(H, \cdot)$ is called *single-power cyclic of period s* .

In an a similar way more complicated hyperstructures can be defined:

$(R, +, \cdot)$ is called *H_v -ring* if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to $(+)$:

$$x(y + z) \cap (xy + xz) \neq \emptyset, (x + y)z \cap (xz + yz) \neq \emptyset, \forall x, y, z \in R.$$

Let $(R, +, \cdot)$ be an *H_v -ring*, $(M, +)$ be a COW *H_v -group* and there exists an external hope

$$\cdot : R \times M \rightarrow P(M) : (a, x) \rightarrow ax$$

such that $\forall a, b \in R$ and $\forall x, y \in M$ we have

$$a(x + y) \cap (ax + ay) \neq \emptyset, (a + b)x \cap (ax + bx) \neq \emptyset, (ab)x \cap a(bx) \neq \emptyset,$$

then M is called an *H_v -module* over F . In the case of an *H_v -field* F , which is defined later, instead of an *H_v -ring* R , then the *H_v -vector space* is defined.

For more definitions and applications on *H_v -structures* one can see the books [4],[6],[16].

Let $(H, \cdot), (H, *)$ be *H_v -semigroups* defined on the same set H . The hope $(\cdot)$ is called *smaller* than the hope $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists an

$$f \in \text{Aut}(H, *) \text{ such that } xy \subset f(x * y), \forall x, y \in H.$$

Then we write $\cdot \leq *$ and we say that $(H, *)$ contains $(H, \cdot)$. If $(H, \cdot)$ is a structure then it is called *basic structure* and $(H, *)$ is called *H_b - structure*.

Theorem 1.1. (*The Little Theorem*). *Greater hopes than the ones which are WASS or COW, are also WASS or COW, respectively.*

This Theorem leads to a partial order on *H_v -structures* and mainly to a correspondence between hyperstructures and posets. Therefore we can obtain an extreme large number of *H_v -structures* just putting more elements on any result. Using the partial ordering with the fundamental relations one can give several definitions to obtain constructions used in several applications [18],[19]:

Let $(H, \cdot)$ be hypergroupoid. We remove $h \in H$, if we consider the restriction of $(\cdot)$ in the set $H - \{h\}$. $\underline{h} \in H$ *absorbs* $h \in H$ if we replace h by $\underline{h}$ and h does not appear in the structure. $\underline{h} \in H$ *merges* with $h \in H$, if we take as product of any $x \in H$ by $\underline{h}$, the union of the results of x with both $h, \underline{h}$, and consider h and $\underline{h}$ as one class with representative $\underline{h}$.

2. The fundamental relations

The main tool to study hyperstructures is the fundamental relation. In 1970 [9] M. Koscas defined in hypergroups the relation β and its transitive closure β^* . This relation connects the hyperstructures with the corresponding classical structures and is defined in *H_v -groups* as well. T. Vougiouklis [12],[13],[16] introduced the γ^* and ϵ^* relations, which are defined, in *H_v -rings* and *H_v -vector spaces*, respectively.

He also named all these relations β^* , γ^* and ϵ^* , Fundamental Relations because they play very important role to the study of hyperstructures especially in the representation theory of them. For similar relations see [1],[3],[4].

Definition 2.1. *The fundamental relations β^* , γ^* and ϵ^* , are defined, in H_v -groups, H_v -rings and H_v -vector space, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively.*

Specifying the above motivation we remark the following: Let $(G, \cdot)$ be a group and R be an equivalence relation (or a partition) in G , then $(G/R, \cdot)$ is an H_v -group, therefore we have the quotient $(G/R, \cdot)/\beta^$ which is a group, the fundamental one. Remark that the classes of the fundamental group $(G/R, \cdot)/\beta^*$ are a union of some of the R -classes. Otherwise, the $(G/R, \cdot)/\beta^*$ has elements classes of G where they form a partition which classes are larger than the classes of the original partition R .*

The way to find the fundamental classes is given by the following:

Theorem 2.2. *Let $(H, \cdot)$ be an H_v -group and denote by U the set of all finite products of elements of H . We define the relation β in H by setting $x\beta y$ iff $\{x, y\} \subset u$ where $u \in U$. Then β^* is the transitive closure of β .*

Now we present a proof for the analogous to the above theorem in the case of an H_v -ring [16]:

Theorem 2.3. *Let $(R, +, \cdot)$ be an H_v -ring. Denote by U the set of all finite polynomials of elements of R . We define the relation γ in R as follows:*

$$x\gamma y \text{ iff } \{x, y\} \subset u \text{ where } u \in U.$$

Then the relation γ^ is the transitive closure of the relation γ .*

Proof. Let $\underline{\gamma}$ be the transitive closure of γ , and denote by $\underline{\gamma}(a)$ the class of the element a . First we prove that the quotient set $R/\underline{\gamma}$ is a ring.

In $R/\underline{\gamma}$ the sum $(\oplus)$ and the product $(\otimes)$ are defined in the usual manner:

$$\underline{\gamma}(a) \oplus \underline{\gamma}(b) = \{\underline{\gamma}(c) : c \in \underline{\gamma}(a) + \underline{\gamma}(b)\},$$

$$\underline{\gamma}(a) \otimes \underline{\gamma}(b) = \{\underline{\gamma}(d) : d \in \underline{\gamma}(a) \cdot \underline{\gamma}(b)\}, \quad \forall a, b \in R.$$

Take $a' \in \underline{\gamma}(a)$, $b' \in \underline{\gamma}(b)$. Then we have

$$a'\underline{\gamma}a \text{ iff } \exists x_1, \dots, x_{m+1} \text{ with } x_1 = a', x_{m+1} = a \text{ and } u_1, \dots, u_m \in U$$

$$\text{such that } \{x_i, x_{i+1}\} \subset u_i, i = 1, \dots, m, \text{ and}$$

$$b'\underline{\gamma}b \text{ iff } \exists y_1, \dots, y_{n+1} \text{ with } y_1 = b', y_{n+1} = b \text{ and } v_1, \dots, v_n \in U$$

$$\text{such that } \{y_j, y_{j+1}\} \subset v_j, j = 1, \dots, n.$$

From the above we obtain

$$\{x_i, x_{i+1}\} + y_1 \subset u_i + v_1, i = 1, \dots, m-1,$$

$$x_{m+1} + \{y_j, y_{j+1}\} \subset u_m + v_j, j = 1, \dots, n.$$

The sums

$$u_i + v_1 = t_i, \quad i = 1, \dots, m-1 \text{ and } u_m + v_j = t_{im+j-1}, \quad j = 1, \dots, n$$

are also polynomials, therefore $t_k \in U$ for all $k \in \{1, \dots, m+n-1\}$.

Now, pick up elements $z_1, \dots, z_{m+n}$ such that

$$z_i \in x_i + y_1, \quad i = 1, \dots, n \text{ and } z_{m+j} \in x_{m+1} + y_{j+1}, \quad j = 1, \dots, n,$$

therefore, using the above relations we obtain $\{z_k, z_{k+1}\} \subset t_k, \quad k = 1, \dots, m+n-1$.

Thus, every element $z_1 \in x_1 + y_1 = a' + b'$ is $\underline{\gamma}$ equivalent to every element $z_{m+n} \in x_{m+1} + y_{n+1} = a + b$. Thus $\underline{\gamma}(a) \oplus \underline{\gamma}(b)$ is a singleton so we can write

$$\underline{\gamma}(a) \oplus \underline{\gamma}(b) = \underline{\gamma}(c) \text{ for all } c \in \underline{\gamma}(a) + \underline{\gamma}(b)$$

In a similar way we prove that

$$\underline{\gamma}(a) \otimes \underline{\gamma}(b) = \underline{\gamma}(d) \text{ for all } d \in \underline{\gamma}(a) \cdot \underline{\gamma}(b)$$

The WASS and the weak distributivity on R guarantee that the associativity and the distributivity are valid for the quotient R/γ^* . Therefore R/γ^* is a ring.

Now let σ be an equivalence relation in R such that R/σ is a ring. Denote $\sigma(a)$ the class of a . Then $\sigma(a) \oplus \sigma(b)$ and $\sigma(a) \otimes \sigma(b)$ are singletons for all $a, b \in R$, i.e. $\sigma(a) \oplus \sigma(b) = \sigma(c)$ for all $c \in \sigma(a) + \sigma(b)$, $\sigma(a) \otimes \sigma(b) = \sigma(d)$ for all $d \in \sigma(a) \cdot \sigma(b)$. Thus we can write, for every $a, b \in R$ and $A \subset \sigma(a)$, $B \subset \sigma(b)$,

$$\sigma(a) \oplus \sigma(b) = \sigma(a + b) = \sigma(A + B), \quad \sigma(a) \otimes \sigma(b) = \sigma(ab) = \sigma(A \cdot B)$$

By induction, we extend these relations on finite sums and products. Thus, for every $u \in U$, we have the relation $\sigma(x) = \sigma(u)$ for all $x \in u$. Consequently

$$x \in \underline{\gamma}(a) \text{ implies } x \in \sigma(a) \text{ for every } x \in R.$$

But σ is transitively closed, so we obtain:

$$x \in \underline{\gamma}(x) \text{ implies } x \in \sigma(a).$$

That means that $\underline{\gamma}$ is the smallest equivalence relation in R such that $R/\underline{\gamma}$ is a ring, i.e. $\underline{\gamma} = \gamma^*$. $\square$

An element is called single if its fundamental class is singleton [16].

Fundamental relations are used for general definitions. Thus we have [13]:

Definition 2.4. An H_v -ring $(R, +, \cdot)$ is called H_v -field if R/γ^* is a field.

Since the algebras are defined on vector spaces, we now present the analogous to the above theorem 2.3, on H_v -vector spaces. The proof is similar.

Theorem 2.5. Let $(V, +)$ be an H_v -vector space over the H_v -field F . Denote by U the set of all expressions consisting of finite hopes either on F and V or the external hope applied on finite sets of elements of F and V . We define the relation ϵ in V as follows:

$$x \epsilon y \text{ iff } \{x, y\} \subset u \text{ where } u \in U.$$

Then the relation ϵ^* is the transitive closure of the relation ϵ .

Definition 2.6. [17],[18],[19] Let $(L, +)$ be an H_v -vector space over the H_v -field $(F, +, \cdot)$, $\phi : F \rightarrow F/\gamma^*$ the canonical map and $\omega_F = \{x \in F : \phi(x) = 0\}$, where 0 is the zero of the fundamental field F/γ . Similarly, let ω_L be the core of the canonical map $\phi' : L \rightarrow L/\epsilon^*$ and denote by the same symbol 0 the zero of L/ϵ^* . Consider the bracket (commutator) hope:

$$[,] : L \times L \rightarrow P(L) : (x, y) \rightarrow [x, y]$$

then L is an H_v -Lie algebra over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.

$$\begin{aligned} [\lambda_1 x_1 + \lambda_2 x_2] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) &\neq \emptyset \\ [x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) &\neq \emptyset, \\ \forall x, x_1, x_2, y, y_1, y_2 \in L, \lambda_1, \lambda_2 \in F \end{aligned}$$

(L2) $[x, x] \cap \omega_L \neq \emptyset, \quad \forall x \in L$

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \emptyset, \quad \forall x, y \in L$

This is a general definition thus one can use special cases in order to face problems in applied sciences.

3. Some classes of H_v -structures

A class of H_v -structures, introduced in [14] is the following:

Definition 3.1. An H_v -structure is called very thin iff all hopes are operations except one, which has all hyperproducts singletons except only one, which is a subset of cardinality more than one. Therefore, in a very thin H_v -structure in a set H there exists a hope $(\cdot)$ and a pair $(a, b) \in H^2$ for which $ab = A$, with $\text{card}A > 1$, and all the other products, with respect to any other hopes (so they are operations), are singletons.

A large class of H_v -structures is the following [20]:

Definition 3.2. Let $(G, \cdot)$ be groupoid (resp., hypergroupoid) and $f : G \rightarrow G$ be a map. We define a hope (∂) , called theta-hope, we write ∂ -hope, on G as follows

$$x\partial y = \{f(x) \cdot y, x \cdot f(y)\}, \quad \forall x, y \in G. \quad (\text{resp. } x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \quad \forall x, y \in G)$$

If $(\cdot)$ is commutative then ∂ is commutative. If $(\cdot)$ is COW, then ∂ is COW.

Let $(G, \cdot)$ be a groupoid (or hypergroupoid) and $f : G \rightarrow P(G) - \{\emptyset\}$ be any multivalued map. We define the (∂) , on G as follows

$$x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \quad \forall x, y \in G$$

Let $(G, \cdot)$ be a groupoid, $f_i : G \rightarrow G, i \in I$, be a set of maps on G . The

$$f_{\cup} : G \rightarrow \mathbf{P}(G) : f_{\cup}(x) = \{f_i(x) | i \in I, \}$$

is the union of $f_i(x)$. We have the union ∂ -hope (∂) , on G if we take $f_{\cup}(x)$. If $\underline{f} \equiv f \cup (id)$, then we have the $b - \partial - hope$.

Motivation for the definition of the theta-hope is the map *derivative* where only the multiplication of functions can be used. The basic property is that if $(G, \cdot)$ is a semigroup then for every f , the (∂) is WASS.

Several results can be obtained using ∂ -hopes [20]:

Consider the group of integers $(\mathbf{Z}, +)$ and $n \neq 0$ be a natural number. Take the map f such that $f(0) = n$ and $f(x) = x, \forall x \in \mathbf{Z} - \{0\}$. Then

$$(\mathbf{Z}, \partial)/\beta^* \cong (\mathbf{Z}_n, +)$$

Theorem 3.3. (a) Take the ring of integers $(\mathbf{Z}, +, \cdot)$ and fix $n \neq 0$ a natural number. Consider the map f such that $f(0) = n$ and $f(x) = x, \forall x \in \mathbf{Z} - \{0\}$. Then $(\mathbf{Z}, \partial_+, \partial.)$, where ∂_+ and $\partial.$ are the ∂ -hopes refereed to the addition and the multiplication respectively, is an H_v -near-ring, with

$$(\mathbf{Z}, \partial_+, \partial.)/\gamma^* \cong (\mathbf{Z}_n).$$

(b) Consider the $(\mathbf{Z}, +, \cdot)$ and $n \neq 0$ a natural. Take the map f such that $f(n) = 0$ and $f(x) = x, \forall x \in \mathbf{Z} - \{n\}$. Then $(\mathbf{Z}, \partial_+, \partial.)$ is an H_v -ring, moreover,

$$(\mathbf{Z}, \partial_+, \partial.)/\gamma^* \cong (\mathbf{Z}_n).$$

Special case of the above is for $n = p$, prime, then $(\mathbf{Z}, \partial_+, \partial.)$ is an H_v -field.

Now we can see theta hopes in H_v -vector spaces and H_v -Lie algebras:

Theorem 3.4. Let $(\mathbf{V}, +, \cdot)$ be an algebra over the field $(F, +, \cdot)$ and $f : V \rightarrow V$ be a map. Consider the ∂ -hope defined only on the multiplication of the vectors $(\cdot)$, then $(\mathbf{V}, +, \partial)$ is an H_v -algebra over F , where the related properties are weak. If, moreover f is linear then we have more strong properties.

Theorem 3.5. Let $(\mathbf{A}, +, \cdot)$ be an algebra over the field F . Take any map $f : A \rightarrow A$, then the ∂ -hope on the Lie bracket $[x, y] = xy - yx$, is defined as follows

$$x\partial y = \{f(x)y - f(y)x, f(x)y - yf(x), xf(y) - f(y)x, xf(y) - yf(x)\}.$$

then $(\mathbf{A}, +, \partial)$ is an H_v -algebra over F , with respect to the ∂ -hopes on Lie bracket, where the weak anti-commutativity and the inclusion linearity is valid.

Remark that if we take the identity map $f(x) = x, \forall x \in A$, then $x\partial y = \{xy - yx\}$, thus we have not a hope and remains the same operation.

Definition 3.6. [11],[16],[17] Let $(G, \cdot)$ be a groupoid then for every $P \subset G, P \neq \emptyset$, we define the following hopes called P -hopes: for all $x, y \in G$

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py), \quad \underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \quad \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The $(G, \underline{P}), (G, \underline{P}_r)$ and $(G, \underline{P}_l)$ are called P -hyperstructures. The most usual case is if $(G, \cdot)$ is semigroup, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$ and $(G, \underline{P})$ is a semihypergroup but we do not know about $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$. In some cases, depending on the choice of P , the $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ can be associative or WASS.

A generalization of P -hopes is the following [7]:

Construction 3.7. Let $(G, \cdot)$ be an abelian group and P any subset of G with more than one elements. We define the hope $\times_P$ as follows:

$$x \times_P y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y | h \in P\} & \text{if } x \neq e \text{ and } y \neq e \\ x \cdot y & \text{if } x = e \text{ and } y = e \end{cases}$$

we call this hope P_e -hope. The hyperstructure $(G, \times_P)$ is an abelian H_v -group.

4. Uniting elements

The *uniting elements* method was introduced by Corsini & Vougiouklis [5] in 1989. With this method one puts in the same class, two or more elements. This leads, through hyperstructures, to structures satisfying additional properties.

The **uniting elements** method is described as follows: Let $\mathbf{G}$ be an algebraic structure and let d be a property, which is not valid. Suppose that d is described by a set of equations; then, consider the partition in $\mathbf{G}$ for which it is put together, in the same partition class, every pair of elements that causes the non-validity of the property d . The quotient by this partition $\mathbf{G}/d$ is an H_v -structure. Then, quotient out the H_v -structure $\mathbf{G}/d$ by the fundamental relation β^* , a stricter structure $(\mathbf{G}/d)\beta^*$ for which the property d is valid, is obtained.

An interesting application of the uniting elements is when more than one properties are desired. The reason for this is some of the properties lead straighter to the classes than others. Therefore, it is better to apply the straightforward classes followed by the more complicated ones. The commutativity is one of the easy applicable properties. Moreover it is clear that the reproductivity property is also easily applicable. One can do this because the following is valid.

Theorem 4.1. [16] Let $(G, \cdot)$ be a groupoid, and

$$F = \{f_1, \dots, f_m, f_{m+1}, \dots, f_{m+n}\}$$

be a system of equations on $\mathbf{G}$ consisting of two subsystems

$$F_m = \{f_1, \dots, f_m\} \text{ and } F_n = \{f_{m+1}, \dots, f_{m+n}\}.$$

Let σ, σ_m be the equivalence relations defined by the uniting elements procedure using the systems F and F_m respectively, and let σ_n be the equivalence relation defined using the induced equations of F_n on the groupoid $\mathbf{G}_m = (\mathbf{G}/\sigma_m)/\beta^*$. Then

$$(\mathbf{G}/\sigma)/\beta^* \cong (\mathbf{G}_m/\sigma_n)/\beta^*$$

Analogous to the above theorem can be proved rings [16]:

Theorem 4.2. *Let $(R, +, \cdot)$ be a ring, and*

$$F = \{f_1, \dots, f_m, f_{m+1}, \dots, f_{m+n}\}$$

be a system of equations on $\mathbf{R}$ consisting of two subsystems

$$F_m = \{f_1, \dots, f_m\} \text{ and } F_n = \{f_{m+1}, \dots, f_{m+n}\}.$$

Let σ, σ_m be the equivalence relations defined by the uniting elements procedure using the systems F and F_m respectively, and let σ_n be the equivalence relation defined using the induced equations of F_n on the ring $\mathbf{R}_m = (\mathbf{R}/\sigma_m)/\gamma^$. Then*

$$(\mathbf{R}/\sigma)/\gamma^* \cong (\mathbf{R}_m/\sigma_n)/\gamma^*$$

i.e. the following diagram is commutative

$$\begin{array}{ccccc} R & \xrightarrow{\rho_m} & R/\sigma_m & \xrightarrow{\phi_m} & R_m \\ \downarrow \rho & & & & \downarrow \rho_n \\ R/\sigma & & & & R_m/\sigma_n \\ \downarrow \phi & & & & \downarrow \phi_n \\ (R/\sigma)/\gamma^* & \xrightarrow{\cong} & & & (R_m/\sigma_n)/\gamma^* \end{array}$$

Where all the maps $\rho, \phi, \rho_m, \phi_m, \rho_n, \phi_n$, are the canonicals.

From the above it is clear that the fundamental structure is very important, mainly if it is known from the beginning. This is the problem to construct hyperstructures with desired fundamental structures. Examples how to obtain special properties as associativity, commutativity, reproductivity can be found in [16],[19].

5. H_v -matrices and representations

H_v -structures are used in Representation Theory of H_v -groups which can be achieved either by generalized permutations or by H_v -matrices [15],[16]. Representations by generalized permutations can be faced by translations. H_v -matrix is called a matrix if has entries from an H_v -ring. The hyperproduct of H_v -matrices is defined in a usual manner. The problem of the H_v -matrix representations is the following:

Definition 5.1. *Let $(H, \cdot)$ be H_v -group, find an H_v -ring R , a set*

$$M_R = \{(a_{ij}) | a_{ij} \in R\}$$

and a map

$$T : H \rightarrow M_R : h \mapsto T(h) \text{ such that } T(h_1 h_2) \cap T(h_1)T(h_2) \neq \emptyset, \forall h_1, h_2 \in H.$$

Then the map T is called H_v -matrix representation.

If the $T(h_1 h_2) \subset T(h_1)(h_2), \forall h_1, h_2 \in H$ is valid, then T is called inclusion representation.

If $T(h_1 h_2) = T(h_1)(h_2) = \{T(h) | h \in h_1 h_2\}, \forall h_1, h_2 \in H$, then T is called good representation and then an induced representation T^* for the hypergroup algebra is obtained.

If T is one to one and good then it is a faithful representation.

In representations of H_v -groups there are two difficulties: To find an H_v -ring or an H_v -field and an appropriate set of H_v -matrices. However more interesting are the small H_v -fields i.e. the results have one or few elements. The single elements, if any exist, are playing a crucial role.

Hopes on any type of ordinary matrices can be defined [8], they are called *helix hopes*. Using several classes of H_v -structures one can face several representations. Some of those classes are as follows [11],[16]:

Definition 5.2. Let $\mathbf{M} = \mathbf{M}_{m \times n}$ be a module of $m \times n$ matrices over a ring $\mathbf{R}$ and $\mathbf{P} = \{P_i : i \in I\} \subseteq \mathbf{M}$. We define, a kind of, a P -hope $\underline{P}$ on $\mathbf{M}$ as follows

$$\underline{P} : \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{P}(\mathbf{M}) : (A, B) \rightarrow \underline{APB} = \{AP_i^t B : i \in I\} \subseteq \mathbf{M}$$

where P^t denotes the transpose of the matrix P .

The hope $\underline{P}$, which is a bilinear map, is a generalization of Rees operation where, instead of one sandwich matrix, a set of sandwich matrices is used. The hope $\underline{P}$ is strong associative and the inclusion distributivity with respect to addition of matrices

$$\underline{AP}(B + C) \subseteq \underline{APB} + \underline{APC} \text{ for all } A, B, C \text{ in } \mathbf{M}$$

is valid. Therefore, $(\mathbf{M}, +, \underline{P})$ defines a multiplicative hyperring on non-square matrices. Multiplicative hyperring means that only the multiplication is a hope.

Definition 5.3. Let $\mathbf{M} = \mathbf{M}_{m \times n}$ be a module of $m \times n$ matrices over R and let us take sets

$$\mathbf{S} = \{s_k : k \in K\} \subseteq R, \quad \mathbf{Q} = \{Q_j : j \in J\} \subseteq \mathbf{M}, \quad \mathbf{P} = \{P_i : i \in I\} \subseteq \mathbf{M}.$$

Define three hopes as follows

$$\underline{S} : R \times \mathbf{M} \rightarrow \mathbf{P}(\mathbf{M}) : (r, A) \rightarrow r\underline{SA} = \{(rs_k)A : k \in K\} \subseteq \mathbf{M}$$

$$\underline{Q}_+ : \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{P}(\mathbf{M}) : (A, B) \rightarrow \underline{AQ}_+ B = \{A + Q_j + B : j \in J\} \subseteq \mathbf{M}$$

$$\underline{P} : \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{P}(\mathbf{M}) : (A, B) \rightarrow \underline{APB} = \{AP_i^t B : i \in I\} \subseteq \mathbf{M}$$

Then $(\mathbf{M}, \underline{S}, \underline{Q}_+, \underline{P})$ is a hyperalgebra over $\mathbf{R}$ called general matrix P -hyperalgebra.

In a similar way a generalization of this hyperalgebra can be defined if one considers an H_v -ring or an H_v -field instead of a ring and using H_v -matrices.

6. From H_v -rings to H_v -fields

Combining the uniting elements procedure with the enlarging theory we can obtain stricter structures or hyperstructures. So enlarging operations or hopes we can obtain more complicated structures.

Theorem 6.1. *In the ring $(\mathbf{Z}_n, +, \cdot)$, with $n = ms$ we enlarge the multiplication only in the product of the special elements $0 \cdot m$ by setting $0 \otimes m = \{0, m\}$ and the rest results remain the same. Then*

$$(\mathbf{Z}_n, +, \otimes) / \gamma^* \cong (\mathbf{Z}_m, +, \cdot)$$

Proof. First we remark that the only expressions of sums and products which contain more than one elements are the expressions which have at least one time the hyperproduct $0 \otimes m$. Adding to this special hyperproduct the element 1, several times we have the $\text{mod } m$ equivalence classes. On the other side, since m is a zero divisor, adding or multiplying elements of the same class the results are remaining in one class, the class obtained by using only the representatives. Therefore, γ^* -classes form a ring isomorphic to $(\mathbf{Z}_m, +, \cdot)$. $\square$

Remark 6.2. *In the above theorem we can enlarge other products as well, for example $2 \cdot m$ by setting $2 \otimes m = \{2, m + 2\}$, then the result remains the same. In this case the elements 0 and 1 remain scalars.*

From the above theorem it is immediate the following result:

Corollary 6.3. *In the ring $(\mathbf{Z}_n, +, \cdot)$, with $n = ps$, where p is a prime number, we enlarge the multiplication only in the product of the special elements $0 \cdot p$ by setting $0 \oplus p = \{0, p\}$ and the rest results remain the same. Then the hyperstructure $(\mathbf{Z}_n, +, \oplus)$ is a very thin H_v -field.*

The above theorem is similar to the Theorem 3.3, but this theorem provides the researchers with H_v -fields appropriate to the representation theory since they may be smaller or minimal hyperstructures.

Remark 6.4. *The above theorem in connection with the Uniting Elements method leads to the fact that in H_v -structure theory it is convenient to equip algebraic structures or hyperstructures with several properties as associativity, commutativity, reproductivity. The important thing is that this equipment can be applied independently of the order of the desired properties. This is crucial point since some properties are easy to be applied, so we can apply them first, and some of the properties are difficult, so we can apply them at the end. For example from an H_v -ring we first go to an H_v -integral domain, by uniting the zero divisors, and then to the H_v -field by reaching the reproductivity.*

7. Some applications

During last decades hyperstructures have a variety of applications in other branches of mathematics and in many other sciences. These applications range from biomathematics -conchology, inheritance- and hadronic physics or on leptons

to mention but a few. The hyperstructures theory is closely related to fuzzy theory; consequently, hyperstructures can now be widely applicable in industry and production, too. In several books and papers [1],[2],[3],[4],[6],[7],[10],[16],[19],[20],[21], one can find numerous applications.

The Lie-Santilli theory on *isotopies* was born in 1970s to solve Hadronic Mechanics problems. Santilli proposed a "lifting" of the n -dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n -dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1999 [7] and they are called *e-hyperfields*. The H_v -fields can give e-hyperfields which can be used in the isotopy theory in applications as in physics or biology.

Definition 7.1. A hyperstructure $(H, \cdot)$ which contain a unique scalar unit e , is called *e-hyperstructure*. In an *e-hyperstructure*, we assume that for every element x , there exists an inverse x^{-1} , i.e. $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$.

Definition 7.2. A hyperstructure $(F, +, \cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called *e-hyperfield* if the following axioms are valid: $(F, +)$ is an abelian group with the additive unit 0 , $(\cdot)$ is WASS, $(\cdot)$ is weak distributive with respect to $(+)$, 0 is absorbing element: $0 \cdot x = x \cdot 0 = 0, \forall x \in F$, there exist a multiplicative scalar unit 1 , i.e. $1 \cdot x = x \cdot 1 = x, \forall x \in F$, and for all $x \in F$ there exists a unique inverse x^{-1} , such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$.

The elements of an e-hyperfield are called *e-hypernumbers*. In the case that the relation: $1 = x \cdot x^{-1} = x^{-1} \cdot x$, is valid, then we say that we have a *strong e-hyperfield*.

Definition 7.3. The Main e-Construction. Given a group $(G, \cdot)$, where e is the unit, then we define in G , a large number of hopes $(\otimes)$ as follows:

$$x \otimes y = \{xy, g_1, g_2, \dots\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, \dots \in G - \{e\}$$

$g_1, g_2, \dots$ are not necessarily the same for each pair (x, y) . Then $(G, \otimes)$ becomes an H_v -group, actually is an H_b -group which contains the $(G, \cdot)$. The H_v -group $(G, \otimes)$ is an *e-hypergroup*. Moreover, if for each x, y such that $xy = e$, so we have $x \otimes y = xy$, then $(G, \otimes)$ becomes a strong *e-hypergroup*.

The proof is immediate since we enlarge the results of the group by putting elements from G and applying the Little Theorem. Moreover one can see that the unit e is a unique scalar and for each x in G , there exists a unique inverse x^{-1} , such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$ and if this condition is valid then we have $1 = x \cdot x^{-1} = x^{-1} \cdot x$. So the hyperstructure $(G, \otimes)$ is a strong e-hypergroup.

The above main e-construction gives an extremely large number of e-hopes. These e-hopes can be used in the several more complicate hyperstructures to obtain appropriate e-hyperstructures.

Example 7.4. Consider the finite-non-commutative quaternion group

$$Q = \{1, -1, i, -i, j, -j, k, -k\}$$

whose multiplication is given by the relations $i^2 = j^2 = -1, ij = -ji = k$. Using this operation one can obtain several hopes which define very interesting e-groups. For example, denoting $\underline{i} = \{i, -i\}, \underline{j} = \{j, -j\}, \underline{k} = \{k, -k\}$ we may define the $(*)$ hope by the table:

*	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	-k	k
i	i	-i	-1	1	k	-k	-j	j
-i	-i	i	1	-1	-k	k	j	-j
j	j	-j	-k	k	-1	1	i	-i
-j	-j	j	k	-k	1	-1	-i	i
k	k	-k	j	-j	-i	i	-1	1
-k	-k	k	-j	j	i	-i	1	-1

$(Q, *)$ is strong e-hypergroup because 1 is scalar unit and the elements -1, i, -i, j, -j, k and -k have unique inverses the elements -1, -i, i, -j, j, -k and k, respectively, which are the inverses in the basic group.

One can see from this example that we can have more strict hopes. The reason we gave the above example is to see that there is a large variety of e-hyperstructures we can construct from given classical structures.

An important new application, which combines hyperstructure theory and fuzzy theory, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis [21]. The suggestion is the following:

Definition 7.5. "In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:

$$0 \text{-----} 1$$

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point s/he feels expresses her/his answer to the specific question".

The use of the Vougiouklis & Vougiouklis bar instead of a Likert scale has several advantages during both the filling-in and the research processing. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm, see [21].

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